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ON BUILDING A RHEOLOGICAL MODEL OF CAVITATING DISPERSIVE LIQUID MEDIA

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It has been shown in [1] that in the process of stretching of pulse-loaded dispersive liquid media (emulsions and dispersions), failure (fragmentation) of the medium is preceded by a stage of unrestricted growth of cavitational bubbles at low concentrations of the disperse phase or by the growth of pores in highly concentrated suspensions (pastes). Therefore a mathematical model of pulsed destruction of dispersive liquid media should also take into account the development of the cavitation process, which is associated with great difficulties because the known mathematical models of bubble suspensions [2–4] are valid only for low-viscosoty liquid matrices in the range of low concentrations of bubbles α_0 without regard for their interaction. And according to [5, 6], even for low-viscosity liquids such as water, as α_0 increases, so does the relaxation time of shear stresses in a bubble medium λ_0, so that $\bar{\lambda}_0 = \lambda_0(\alpha_0)/\lambda_0(\alpha_0 = 0) \simeq 10^4$ holds even at $\alpha_0 = 0.8$, and the medium acquires viscoelastic properties. But all concentrated emulsions and suspensions (including those with low-viscosity matrices) have sufficiently long relaxation times of shear stresses λ_1^0 for emulsions and λ_2^0 for suspensions even in the initial state at $\alpha_0 = 0$ [1], i.e., before the process of medium extension starts. Hence, as α_0 grows, λ_1^0 and λ_2^0 must increase, and the media being stretched must acquire ever more expressed elastic properties.

Thus, in a mathematical model of the dynamic destruction of dispersive liquid media it is essential to take into account the evolution of their viscoelastic properties in the process of media morphology transformation connected with cavitation processes. In the present paper a macrorheological approach is used to create a physico-mathematical model of the process under study. Within the framework of this approach an attempt was made to build a mechanical model of cavitating dispersive liquid media (DLM) which holds in the range of α_0 from zero to values corresponding to the formation of cellular structures in a disperse medium ($\alpha_0 > 0.9$), and to discuss the problem of the effect exerted by the cavitational bubbles present in DLM on its viscoelastic properties.

1. To build a rheological model that cavitates with monotonic stretching of DLM, it is necessary to analyze the evolution of the medium's morphology in the stretching process and to create a mechanical model that is adequate at every stage of the process.

Let μ_0 be the effective shearing viscosity of a medium, and G_1 and μ_1, respectively, the shear elastic modulus and the shear viscosity of the disperse phase (solid in suspensions and liquid in emulsions). If, before the DLM is stretched, it contains no bubbles with accuracy to cavitation nuclei, i.e., $\alpha_0 \approx 0$, then this medium can be described by a mechanical model generally accepted in rheology [7] (Fig. 1a), i.e., by connecting in series viscous element μ_0 corresponding to the medium's fluidity with the Voigt node $G_1|\mu_1$, which represents a parallel connection of viscous element μ_1 with elastic element G_1 and corresponds to the viscoelastic properties of dispersed elements with nonzero compressibility.

The flow of a liquid matrix is accompanied by deformation of the dispersed elements (in an emulsion the shear elasticity of droplets is determined by the interfacial tension on the interface of the droplet and the matrix). According to this mechanical scheme, in view of the series connection of viscous element μ_0 and node $G_1|\mu_1$, the strains of the liquid matrix and of the viscoelastic dispersed elements are added up, while the stresses in the matrix and those in the dispersed elements are equal.

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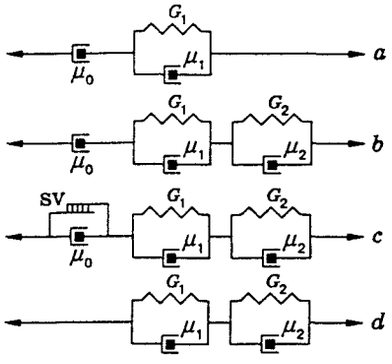

Fig. 1

If a tensile stress is applied to such a medium, then in this medium, as has been noted, cavitation bubbles or pores begin to grow out of cavitation nuclei contained in the liquid matrix and at the interface with the disperse phase, and in the medium there appear dispersed elements that are new in phase. And since bubbles introduce additional viscosity and elasticity into the medium (due to the elasticity of their form), the mechanical scheme of the medium must be supplemented by a second Voigt node $\mu_2|G_2$ (Fig. 1b) corresponding to the viscoelastic properties of bubbles.

But if α_0 reaches and then exceeds the concentration of the boundary packing of bubbles α_{0*}, then the medium must enter a qualitatively new rheological state. Indeed, at $\alpha_0 = \alpha_{0*}$ all bubbles come into contact with one another, and further, at $\alpha_0 > \alpha_{0*}$, due to their consolidation the medium loses its property of fluidity, at least under small shear strains, and becomes into a foamy frame. According to [8], the static shear modulus of dry foam ($\alpha_0 > 0.95$) can be estimated by the formula $G \simeq \sigma S_0/3$, where σ is the surface tension coefficient of the matrix liquid and S_0 is the specific area of the foam cells (in the case of a water matrix for dry foam $G \sim 10^2$ Pa).

In order to have a clearer idea of the rheological characteristics of a disperse medium at $\alpha_0 > 0.9$, i.e., when cells of a foamy structure begin to be formed in this medium, let us analyze the results of the work reported in [9], where a theoretical model was used to study the behavior of stationary dry foams (having a homogeneous liquid matrix) under shear strains. If a simple shear tension τ is instantaneously applied to a sample of dry foam with cells of a hexagonal structure (Fig. 2a; the control cells are marked with hatching and dots), then the medium will behave as follows (Figs. 2 and 3 are borrowed from [9]). At first (at $\tau = 0$) the medium has a honeycomb structure with energetically stable trihedral Plato nodes (Fig. 2a'). Further, as τ increases to take the values corresponding to marks 2 and 3 in Fig. 3a, there occurs an elastic shear strain of the cells, as shown in Figs. 2b and 3 (in Fig. 3 ε is the shear strain). Finally, when τ reaches a certain threshold value ($\tau = \tau^*$), there occurs a coalescence of cells to form electrically unstable cells (Fig. 2d) with tetrahedral Plato nodes (Fig. 2d'). If in this case the capillary number $\mathrm{Ca} = \sqrt{3}\,(1 - \alpha_0)\,\mu_0 a\dot{\varepsilon}/4\sigma$ (a is the characteristic size of the cells) is less than the threshold value $\mathrm{Ca}^* \approx 2.7 \cdot 10^{-7}$, then, as shown in Figs. 2 and 3a, where $\mathrm{Ca} = 10^{-7}$, $\alpha_0 = 0.98$, the system spontaneously enters a state with a lower level of free energy. This state is reached by restructuring the system morphology so that hexagonal cells with trihedral Plato nodes are formed again (Fig. 2e). The process is accompanied by load relieving: the stress drops to a value somewhat lower than zero (Fig. 3a, point 5). Owing to this there occurs a relative shear in the cells marked by hatching (Fig. 2a $\rightarrow$ Fig. 2e). With further cell stretching the process is repeated (Fig. 3a), i.e., the graph of $\tau(\varepsilon)$ is periodic in character.

But if $\mathrm{Ca} > \mathrm{Ca}^*$, then, according to the data of [9], the pulsations in the graph of $\tau(\varepsilon)$ are smoothed (Fig. 3b,c). This is due to the fact that in the process of shear straining of cells at $\mathrm{Ca} > \mathrm{Ca}^*$ there occurs no coalescence of their boundaries or restoration of their hexagonal shape with trihedral Plato nodes, as opposed to the cases when $\mathrm{Ca} < \mathrm{Ca}^*$ (Fig. 2d,e). To put it differently, at $\mathrm{Ca} > \mathrm{Ca}^*$ there occurs continuous relative

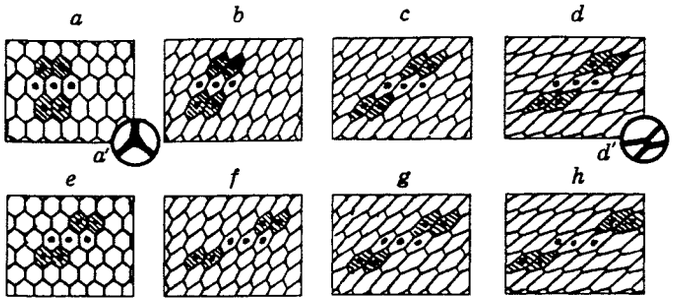

Fig. 2

shifting of cells resembling the plastic strain in metals when the tangential stress exceeds the static limit of fluidity, and in the medium there occurs above-barrier slipping of dislocations.

Thus, a detailed analysis of the result obtained in [9] makes it possible to draw the following conclusion. If in a liquid medium which is being stretched the concentration of cavitation bubbles $\alpha_0 > 0.9$, then at shear stresses less than the critical value of τ^* the medium loses its property of fluidity: there occurs a peculiar kind of cell consolidation in the liquid matrix, and the medium behaves like a solid frame. But if $\tau > \tau^*$ then, because of relative shifting of the cells (Fig. 3), the medium enters a plastofluid state similar to that of a Bingham body.

With regard to the above analysis, the mechanical model of a DLM (emulsion or suspension) (Fig. 1b) in which, due to its stretching, the cavitational bubbles grow monotonically to reach the state of a foamy structure ($\alpha_0 > 0.9$), should be supplemented by a plastofluid St. Venant element [7] connected in parallel with viscous element μ_0 (Fig. 1c). At shear stresses smaller than τ^*, this medium possesses a property of a solid: element SV of the scheme blocks the fluidity of viscous element μ_0, and the mechanical model is reduced to the scheme presented in Fig. 1d. If the shear stress exceeds τ^*, then the medium's structure, i.e., its "rigid frame" of foam is destroyed (element μ_0 is unblocked), and the medium behaves like a viscoelastic flowing liquid body (Fig. 1b). In this case the liquid matrix elasticity at real shear strain rates is not taken into account.

In its final form (Fig. 1c) the mechanical model operates as follows. In their initial state the cavitational nuclei, due to their small sizes (10^{-3}–10^{-4} cm), can be considered unstrained (in the sense of shear strains), i.e., at $\alpha_0 \to 0$ $G_2 \to \infty$ and in the general model of the medium (Fig. 1c) the node $\mu_2|G_2$ can be disregarded. In this case clearly the medium possesses the property of fluidity at any τ, i.e., due to the absence of bubbles the St. Venant limiting stress τ^* for the medium is equal to zero, and the mechanical model of Fig. 1c degenerates into a model of an Oldroyd body (Fig. 1a), whose rheological characteristics depend on the concentration and the viscoelastic properties of dispersed elements.

With increasing α_0, G_2 decreases, which initiates the node $\mu_2|G_2$ in the mechanical model (Fig. 1b). After α_0 reaches the value of the boundary packing of bubbles α_{0*}, at $\alpha_0 > \alpha_{0*}$ the bubbles begin to come into mutual contact, i.e., to form a foamy nuclear frame [1] and block the medium's fluidity under small shear stresses. But if τ exceeds the threshold stress τ^*, which, as can be concluded from Fig. 3, depends on α_0, then the medium restores its plastofluid property via cell shifting. Accordingly, in the mechanical model of the medium at $\alpha_0 > \alpha_{0*}$ there appears a St. Venant element SV (Fig. 1c) controlling the limiting shear stress (blocking the medium's fluidity) by the law

$$\tau^*(\alpha_0) = \begin{cases} 0 & \text{at} \quad \alpha_0 < \alpha_{0*}, \\ f(\alpha_0) & \text{at} \quad \alpha_0 \geqslant \alpha_{0*}. \end{cases} \tag{1.1}$$

2. Using the mechanical model of a cavitating DLM, one can derive (by the methods of theoretical rheology [7, 10]) a rheological equation relating, in a differential form, stresses to strains via the main rheological constants of the medium, i.e., at $\mu_i = \text{const}$, $G_i = \text{const}$ for any fixed volume concentration of bubbles α_0.

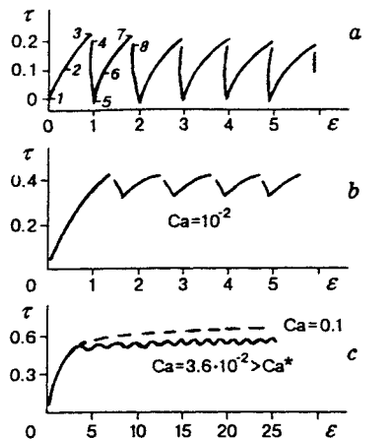

Fig. 3

Let P_{ij} and d_{ij} be the tensors of the medium's stresses and strains, respectively, and σ_{ij} and e_{ij}, the deviators of these tensors. Then, since the mechanical model of the medium (Fig. 1c) represents a series connection of three two-element nodes ($\mu_0|\mathrm{SV}$, $\mu_1|G_1$, and $\mu_2|G_2$), the resulting strain tensor deviator e_{ij} is equal to the sum of the strain tensor deviators: e^0_{ij} of the node $\mu_0|\mathrm{SV}$, e'_{ij} of the node $\mu_1|G_1$, and e''_{ij} of the node $\mu_2|G_2$. In this case the stress tensor deviator σ_{ij} will be the same for all the three nodes. In the case of the mechanical model of a Bingham body, by definition [7, 10], if the generalized shear stress (tangential stress intensity) $\tau_i = [(1/2)\,\sigma_{ij}\sigma_{ij}]^{1/2}$ is less than Θ, then the body will be absolutely solid, i.e., $e_{ij} = 0$. But if $\tau_i = \Theta$, then in the medium there occurs a plastoviscous flow. Since in the node $\mu_0|\mathrm{SV}$ the stress tensor deviators in parallel branches of the St. Venant plastofluid element and of the viscous Newtonian liquid add up together, the resulting stress tensor deviator of this node is written in the form

$$\sigma_{ij} = 2\eta_* \dot{e}^0_{ij} + 2\mu_0 \dot{e}^0_{ij}, \tag{2.1}$$

where η_* (effective viscosity coefficient of the SV viscoplastic element) is a scalar variable [7].

Similarly, for nodes $\mu_1|G_1$ and $\mu_2|G_2$ containing elastic (Hookian) and viscous (Newtonian) elements connected in parallel the following can be written accordingly

$$\sigma_{ij} = 2G_1 e'_{ij} + 2\mu_1 \dot{e}'_{ij} \tag{2.2}$$

and

$$\sigma_{ij} = 2G_2 e''_{ij} + 2\mu_2 \dot{e}''_{ij}. \tag{2.3}$$

Since, according to the mechanical model of the medium (Fig. 1c),

$$e_{ij} = e^0_{ij} + e'_{ij} + e''_{ij}, \tag{2.4}$$

we rewrite (2.1)–(2.3) in operator form, substitute the values of their e^0_{ij}, e'_{ij}, and e''_{ij} into (2.4), and obtain

$$e_{ij} = \frac{\sigma_{ij}}{\{2\,(\eta_* + \mu_0)\,\frac{\partial}{\partial t}\}} + \frac{\sigma_{ij}}{\{2\,(G_1 + \mu_1\,\frac{\partial}{\partial t})\}} + \frac{\sigma_{ij}}{\{2\,(G_2 + \mu_2\,\frac{\partial}{\partial t})\}} \tag{2.5}$$

or, after transformation,

$$G_1G_2\sigma_{ij} + [G_1\mu_2 + G_2\mu_1 + G_2\mu_0 + G_1\mu_0 + (G_1 + G_2)\,\eta_*]\,\dot{\sigma}_{ij} + [\mu_1\mu_2 + \mu_0\mu_2 + \mu_0\mu_1 + (\mu_1 + \mu_2)\,\eta_*]\,\ddot{\sigma}_{ij}$$
$$= 2G_1G_2\,(\mu_0 + \eta_*)\,\dot{e}_{ij} + +2\,[\mu_0\mu_2G_1 + \mu_0\mu_1G_2 + (\mu_1G_2 + \mu_2G_1)\,\eta_*]\ddot{e}_{ij} + 2\mu_1\mu_2\,(\mu_0 + \eta_*)\,\dddot{e}_{ij}\,. \tag{2.6}$$

Considering that, according to the principle of passage of a plastic SV element through a limiting shear stress Θ, a viscous element connected in parallel with this plastic element does not prevent the plastic element

from reaching the value Θ [10], and assuming that in (2.6) $\mu_0 = 0$, we have

$$\{2[G_1G_2\dot{e}_{ij} + (\mu_1G_1 + \mu_2G_2)\ddot{e}_{ij} + \mu_1\mu_2\dddot{e}_{ij}] - (G_1 + G_2)\dot{\sigma}_{ij} - (\mu_1 + \mu_2)\ddot{\sigma}_{ij}\}\eta_* = G_1G_2\sigma_{ij} + (G_1\mu_2 + G_2\mu_1)\dot{\sigma}_{ij} + \mu_1\mu_2\ddot{\sigma}_{ij}. \tag{2.7}$$

According to the Mises equation [11], plastic strains in a plastically strained body appear when the generalized tangential shear stress reaches the limiting value

$$\tau_i = \left(\frac{1}{2}\sigma_{ij}\sigma_{ij}\right)^{1/2} = \Theta. \tag{2.8}$$

If we multiply (2.7) by σ_{ij} and substitute $\sigma_{ij}\sigma_{ij} = 2\Theta^2$ from (2.8) into (2.7), then, taking into account that $\Theta(\alpha_0 = \text{const}) = \text{const}$ and, hence, according to (2.8), $\sigma_{ij}\dot{\sigma}_{ij} = 0$, we have

$$\{2[G_1G_2\dot{e}_{ij} + (\mu_1G_2 + \mu_2G_1)\ddot{e}_{ij} + \mu_1\mu_2\dddot{e}_{ij}]\sigma_{ij} - (\mu_1 + \mu_2)\ddot{\sigma}_{ij}\sigma_{ij}\}\eta_* = 2\Theta^2G_1G_2 + \mu_1\mu_2\ddot{\sigma}_{ij}\sigma_{ij}. \tag{2.9}$$

Further, since $\sigma_{ij}\dot{\sigma}_{ij} = 0$, then $\sigma_{ij}\ddot{\sigma}_{ij} = d(\sigma_{ij}\dot{\sigma}_{ij})/dt - \dot{\sigma}_{ij}\dot{\sigma}_{ij} = -\dot{\sigma}_{ij}\dot{\sigma}_{ij}$, so $\sigma_{ij}\sigma_{ij}\ddot{\sigma}_{ij} = -(\sigma_{ij}\dot{\sigma}_{ij})\dot{\sigma}_{ij} = 0$. Taking this into account and multiplying (2.9) by σ_{ij} we obtain

$$\eta_* = \frac{\Theta^2}{[\dot{e}_{ij} + (\bar{\lambda}_1 + \bar{\lambda}_2)\ddot{e}_{ij} + \bar{\lambda}_1\bar{\lambda}_2\dddot{e}_{ij}]\sigma_{ij}}, \tag{2.10}$$

where $\bar{\lambda}_1 = \mu_1/G_1$, $\bar{\lambda}_2 = \mu_2/G_2$ are the characteristic temporal parameters of the medium. Dividing (2.6) by G_1G_2, solving this equation for η_*, substituting into it Eq. (2.10) for η_*, and regrouping the terms we obtain a rheological equation for a DLM containing cavitation bubbles in the range of fixed concentrations from cavitation nuclei to a cellular solidlike structure:

$$\sigma_{ij} + \left[\bar{\lambda}_1 + \bar{\lambda}_2 + \mu_0\left(\frac{1}{G_1} + \frac{1}{G_2}\right)\right]\dot{\sigma}_{ij} + \left[\bar{\lambda}_1\bar{\lambda}_2 + \mu_0\left(\frac{\bar{\lambda}_2}{G_1} + \frac{\bar{\lambda}_1}{G_2}\right)\right]\ddot{\sigma}_{ij}$$
$$+ \left[\frac{(\frac{1}{G_1} + \frac{1}{G_2})\dot{\sigma}_{ij} + (\frac{\bar{\lambda}_1}{G_2} + \frac{\bar{\lambda}_2}{G_1})\ddot{\sigma}_{ij}}{\dot{e}_{ij} + (\bar{\lambda}_1 + \bar{\lambda}_2)\ddot{e}_{ij} + \bar{\lambda}_1\bar{\lambda}_2\dddot{e}_{ij}} - 2\right]\frac{\Theta^2}{\sigma_{ij}} = 2\mu_0[\dot{e}_{ij} + (\bar{\lambda}_1 + \bar{\lambda}_2)\ddot{e}_{ij} + \bar{\lambda}_1\bar{\lambda}_2\dddot{e}_{ij}]. \tag{2.11}$$

As noted earlier, this equation holds on the condition that all of its coefficients (μ_0, μ_1, μ_2, G_1, G_2, and Θ) are time-independent. This can be the case only when in the process of straining α_0 remains constant. (The values of the volume concentration of bubbles α_1 and solid particles α_2, with the mass of the medium remaining the same, will always be constant.) But since the strain tensor deviator e_{ij} contains only a change in the shape of the medium with its volume remaining constant, Eq. (2.11) describes the medium's strain process without any changes in α_0, and, hence, with strains described by Eq. (2.11), the rheological coefficients remain constant.

Equation (2.11) can be reduced to an equation for purely shear strains:

$$\tau + \left[\bar{\lambda}_1 + \bar{\lambda}_2 + \mu_0\left(\frac{1}{G_1} + \frac{1}{G_2}\right)\right]\dot{\tau} + \left[\bar{\lambda}_1\bar{\lambda}_2 + \mu_0\left(\frac{\bar{\lambda}_2}{G_1} + \frac{\bar{\lambda}_1}{G_2}\right)\right]\ddot{\tau}$$
$$+ \left[\frac{(\frac{1}{G_1} + \frac{1}{G_2})\dot{\tau} + (\frac{\bar{\lambda}_1}{G_2} + \frac{\bar{\lambda}_2}{G_1})\ddot{\tau}}{\dot{\varepsilon} + (\bar{\lambda}_1 + \bar{\lambda}_2)\ddot{\varepsilon} + \bar{\lambda}_1\bar{\lambda}_2\dddot{\varepsilon}} - 1\right]\frac{\tau^{*2}}{\tau} = \mu_0[\dot{\varepsilon} + (\bar{\lambda}_1 + \bar{\lambda}_2)\ddot{\varepsilon} + \bar{\lambda}_1\bar{\lambda}_2\dddot{\varepsilon}]. \tag{2.12}$$

Here τ^* meets condition (1.1); τ and ε are the stress and the strain of a pure shear, respectively; the dependence $\tau^* = f(\alpha_0 > \alpha_{0*})$ can be determined experimentally.

It should be noted that in the case of suspensions, for G_1 and μ_0 to be independent of the medium's strain rate, the value of α_2, according to [12], should not exceed 0.35. In the case of emulsions no dependence of μ_0 or G_1 on the medium strain rate is observed at any value of α_1 [13]. The dependence of μ_2 and G_2 on the strain rate requires special experimental research. Therefore we will further assume the shear strain rate in media with $\alpha_0 > 0$ to be sufficiently low.

3. Consider some partial solutions of Eq. (2.12) making it possible to analyze the reaction of DLM to various disturbances of its state.

3.1. Let the volume concentration of bubbles $\alpha_0 > \alpha_{0*}$, and let them form in the medium a cellular frame, i.e., $\tau^* > 0$ (Fig. 1c). To conveniently analyze the behavior of the medium under study, here and further we will use, as is the convention in rheology, a step disturbance:
with shear stress

$$\tau(t) = \tau_0 [U(t)], \quad [U(t)] = \begin{cases} 0, & t < 0, \\ 1, & t \geqslant 0 \end{cases} \tag{3.1}$$

or with shear strain

$$\varepsilon(t) = \varepsilon_0 [U(t)], \quad [U(t)] = \begin{cases} 0, & t < 0, \\ 1, & t \geqslant 0. \end{cases} \tag{3.2}$$

(A) The case when $\tau = \tau_0 [U(t)] > \tau^*$: a structured DLM possesses plastic-viscoelastic properties, and Eq. (2.12) is reduced to the form

$$\ddot{\varepsilon} + \frac{\lambda_2}{æ_2}\ddot{\varepsilon} + \frac{1}{æ_2}\dot{\varepsilon} = \left(1 - \frac{\tau^{*2}}{\tau_0^2}\right)\frac{\tau_0}{\mu_0 æ_2}, \tag{3.3}$$

where $æ_2 = \bar{\lambda}_1\bar{\lambda}_2$; $\lambda_2 = \bar{\lambda}_1 + \bar{\lambda}_2$.

Under the initial conditions $\varepsilon(t=0) = \varepsilon_0$, $\dot{\varepsilon}(t=0) = \varepsilon_0$, and $\ddot{\varepsilon}(t=0) = \ddot{\varepsilon}_0$ Eq. (3.3) has a general solution

$$\varepsilon(t) = \left[\varepsilon_0 + (\bar{\lambda}_1 + \bar{\lambda}_2)\dot{\varepsilon}_0 + \bar{\lambda}_1\bar{\lambda}_2\ddot{\varepsilon}_0 - (\bar{\lambda}_1 + \bar{\lambda}_2)\left(1 - \frac{\tau^{*2}}{\tau_0^2}\right)\frac{\tau_0}{\mu_0}\right] + \left(1 - \frac{\tau^{*2}}{\tau_0^2}\right)\frac{\tau_0}{\mu_0}t$$

$$+ \frac{\ddot{\varepsilon}_0\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2 e^{-t/\bar{\lambda}_2} - \bar{\lambda}_1 e^{-t/\bar{\lambda}_1})}{\bar{\lambda}_1 - \bar{\lambda}_2} + \frac{[\dot{\varepsilon}_0 - (1 - \tau^{*2}/\tau_0^2)\tau_0/\mu_0]}{\bar{\lambda}_1 - \bar{\lambda}_2}(\bar{\lambda}_2^2 e^{-t/\bar{\lambda}_2} - \bar{\lambda}_1^2 e^{-t/\bar{\lambda}_1}). \tag{3.4}$$

Here the second term on the right-hand side [i.e., $(1 - \tau^{*2}/\tau_0^2)\tau_0 t/\mu_0$] is a result of the medium's fluidity (mutual "slipping" of cells, Figs. 2 and 3), which is the greater, the greater the ratio τ_0/τ^* and the smaller the effective viscosity μ_0; the third and the fourth terms on the right-hand side describe the process of the elastic strain delay of a DLM with relaxation times $\bar{\lambda}_1$ and $\bar{\lambda}_2$. The strain lags are due to the presence in the medium of elastic solid-disperse elements or droplets ($\bar{\lambda}_1$) and bubbles or cells ($\bar{\lambda}_2$). Consequently, according to (3.4), at $\tau_0 - \tau^* \to +0$ the intensity of plastic-viscous flow is completely damped, and the medium degenerates into a viscoelastic Voigt body (Fig. 1d).

(B) If $\tau < \tau^*$, then element SV of node SV$|\mu_0$ is, by definition, absolutely solid, it blocks the element μ_0 corresponding to a viscous flow, and the medium loses its property of fluidity and is converted into a viscoelastic "solidlike" generalized Voigt body (Fig. 1d), i.e., a rigid cellular frame. But if a pure shear stress $\tau_0 < \tau^*$ is applied to this medium, then, in view of the uniformity of nodes $\mu_1|G_1$ and $\mu_2|G_2$, the shear strains in them will add up, and at $\varepsilon_0 = 0$, $\dot{\varepsilon}_0 = 0$ the total shear strain, according to [14], will have the form

$$\varepsilon(t) = \tau_0 [J_1(1 - e^{-t/\bar{\lambda}_1}) + J_2(1 - e^{-t/\bar{\lambda}_2})], \tag{3.5}$$

where $J_i = 1/G_i$ $(i = 1, 2)$ is the shear compliance. Thus, in the medium the shear strain will grow with a lag characterized by constant times $\bar{\lambda}_1$ and $\bar{\lambda}_2$ dependent on the viscoelastic properties of the solid particles or droplets ($\bar{\lambda}_1$) and bubbles or cells ($\bar{\lambda}_2$). According to (3.5), $\varepsilon(t) \to \tau_0(J_1 + J_2)$ for $t \to \infty$.

(C) Let $\varepsilon(t) = \varepsilon_0 [U(t)]$, where $\varepsilon_0 < \tau^*(J_1 + J_2)$ is the instantaneous strain of the DLM under conditions (3.2). Since in this case element SV remains absolutely solid by definition, we assume that in (2.5) $\eta_* \to \infty$ and obtain

$$e_{ij} = \sigma_{ij}/\left\{2\left(G_1 + \mu_1\frac{\partial}{\partial t}\right)\right\} + \sigma_{ij}/\left\{2\left(G_2 + \mu_2\frac{\partial}{\partial t}\right)\right\}$$

and (2.12) reduces to the form

$$\frac{\mu_1+\mu_2}{G_1G_2}\dot{\tau}+\left(\frac{1}{G_1}+\frac{1}{G_2}\right)\tau=æ_2\ddot{\varepsilon}+\lambda_2\dot{\varepsilon}+\varepsilon,$$

or, since $\varepsilon(t)=\varepsilon_0[U(t)]$,

$$\dot{\tau}+\frac{G_1+G_2}{\mu_1+\mu_2}\tau=\frac{G_1G_2}{\mu_1+\mu_2}\varepsilon_0.$$

The solution of this equation is exponential in character:

$$\tau=\tau_0 e^{-t/\tilde{\lambda}}+\frac{\varepsilon_0}{J_1+J_2}(1-e^{-t/\tilde{\lambda}}). \tag{3.6}$$

Here $\tilde{\lambda}=(\mu_1+\mu_2)/(G_1+G_2)$ is the relaxation time of shear stress in the DLM, which is dependent on the rheological parameters of both the dispersed elements and the bubbles. But if the dispersed elements are absolutely solid, i.e., $G_1\to\infty$, then from (3.6) we have $\tau=G_2\varepsilon_0$; in this case the DLM behaves like an elastic Hooke body.

3.2. The volume concentration of bubbles $\alpha_0<\alpha_{0*}$, i.e., according to (1.1), $\tau^*=0$: the DLM is a fluid medium under any shear stress, and (2.12) reduces to the form

$$\tau+\lambda_1\dot{\tau}+æ_1\ddot{\tau}=\mu_0(\dot{\varepsilon}+\lambda_2\ddot{\varepsilon}+æ_2\dddot{\varepsilon}), \tag{3.7}$$

where $\lambda_1=(\mu_0+\mu_1)/G_1+(\mu_0+\mu_2)/G_2$; $æ_1=(\mu_0\mu_1+\mu_0\mu_2+\mu_1\mu_2)/G_1G_2$.

Consider two modes of disturbance of such a medium.

(A) If $\tau=\tau_0[U(t)]$, then (3.7) reduces to the form

$$\dddot{\varepsilon}+\frac{\lambda_2}{æ_2}\ddot{\varepsilon}+\frac{1}{æ_2}\dot{\varepsilon}=\frac{\tau_0}{\mu_0æ_2}$$

and under the initial conditions $\varepsilon(t=0)=\varepsilon_0$, $\dot{\varepsilon}(t=0)=\dot{\varepsilon_0}$, and $\ddot{\varepsilon}(t=0)=\ddot{\varepsilon}_0$ it has the solution

$$\varepsilon(t)=(\varepsilon_0+\lambda_2\dot{\varepsilon}_0+æ_2\ddot{\varepsilon}_0-\lambda_2\frac{\tau_0}{\mu_0})+\frac{\tau_0}{\mu_0}t-\frac{\bar{\lambda}_1^2(\bar{\lambda}_2\ddot{\varepsilon}_0+\dot{\varepsilon}_0-\frac{\tau_0}{\mu_0})}{\bar{\lambda}_1-\bar{\lambda}_2}e^{-t/\bar{\lambda}_1}+\frac{\bar{\lambda}_2^2(\bar{\lambda}_1\ddot{\varepsilon}_0+\dot{\varepsilon}_0-\frac{\tau_0}{\mu_0})}{\bar{\lambda}_1-\bar{\lambda}_2}e^{-t/\bar{\lambda}_2}. \tag{3.8}$$

Hence it follows that on loading a DLM containing a fixed concentration of unconsolidated bubbles ($\alpha_0<\alpha_{0*}$) with disturbance (3.1) the medium's shear strain will be determined by the medium's fluidity (the term τ_0t/μ_0), which is more intense than in case (A) of Section 3.1, as well as by the elastic strain lag due to the presence in the medium of dispersed elements and bubbles [the third and forth terms on the right-hand side of (3.8)]. The lag times $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are determined by the rheological constants of the disperse phase (μ_1, G_1) and of the bubbles (μ_2, G_2).

(B) Let $\varepsilon=\varepsilon_0[U(t)]$. Substituting this function into (3.7) we obtain the equation

$$\ddot{\tau}+\frac{\lambda_1}{æ_1}\dot{\tau}+\frac{1}{æ_1}\tau=0,$$

the solution of which for $\tau(t=0)=\tau_0$ and $\dot{\tau}(t=0)=\dot{\tau}_0$ has the form

$$\tau(t)=\left[\frac{\tau_0}{2}\left(1+\frac{1}{\xi_1}\right)+\frac{\xi_2}{2\xi_1}\dot{\tau}_0\right]e^{-t/\tilde{\lambda}_1}-\left[\frac{\tau_0}{2}\left(\frac{1}{\xi_1}-1\right)+\frac{\xi_2}{2\xi_1}\dot{\tau}_0\right]e^{-t/\tilde{\tilde{\lambda}}_1}, \tag{3.9}$$

where

$$\xi_1=\sqrt{|1-4æ_1/\lambda_1^2|},\quad \xi_2=2æ_1/\lambda_1,\quad \tilde{\lambda}_1=\xi_2/(1-\xi_1),\quad \tilde{\tilde{\lambda}}_1=\xi_2/(1+\xi_1). \tag{3.10}$$

Using (3.10) we can show that

$$\tilde{\lambda}_1=\lambda_1'\left[1+\frac{\lambda_1''}{\lambda_1'}\left(1-\frac{æ_1}{\lambda_1\lambda_1''}\right)+\frac{æ_1^2}{\lambda_1^3\lambda_1'}\left(1-\frac{2æ_1}{\lambda_1^2}\right)+\ldots\right]. \tag{3.11}$$

Here

$$\lambda_1' = \frac{\mu_0 + \mu_1}{G_1}, \quad \lambda_1'' = \frac{\mu_0 + \mu_2}{G_2}, \quad \frac{æ_1}{\lambda_1 \lambda_1''} < 1, \quad \frac{2æ_1}{\lambda_1^2} < 1. \tag{3.12}$$

From (3.9) it follows that in a DLM containing unconsolidated bubbles under disturbances of the (3.2) type shear stresses relax according to an exponential law with constant relaxation times $\tilde{\lambda}_1$ and $\tilde{\tilde{\lambda}}_1$ dependent on the rheological constants of the liquid phase, disperse phase, and bubbles.

If $\alpha_0 \to 0$ on retention of a countable concentration of bubbles, then, as noted earlier, $G_2 \to \infty$, so $æ_1 \to 0$, $æ_2 \to 0$, $\lambda_2 \to \bar{\lambda}_1$, $\lambda_1 \to \lambda_1'$, and (3.7) reduces to the Oldroyd equation [14]

$$\tau + \lambda_1' \dot{\tau} = \mu_0 (\dot{\varepsilon} + \bar{\lambda}_1 \ddot{\varepsilon}), \tag{3.13}$$

which describes the rheological properties of emulsions and suspensions. It can readily be verified that $G_2 \to \infty$ for $\alpha_0 \to 0$, and solutions (3.8) and (3.9) of Eq. (3.7) for a DLM with bubbles also degenerate into corresponding solutions of Eq. (3.13):

$$\varepsilon(t) = \varepsilon_0 + \frac{\tau_0}{\mu_0} t - \bar{\lambda}_1 \left(\frac{\tau_0}{\mu_0} - \dot{\varepsilon}_0 \right) (1 - e^{-t/\bar{\lambda}_1}), \quad \tau(t) = \tau_0 e^{-t/\lambda_1'}.$$

Since in solution (3.9) $\tilde{\lambda}_1 > \tilde{\tilde{\lambda}}_1$, the second term on its right-hand side decreases with time faster than the first term, and thus the character of the relaxation τ will be determined by the time constant $\tilde{\lambda}_1$. And according to (3.11) $\tilde{\lambda}_1 > \lambda_1'$; hence, if bubbles with a volume concentration of bubbles α_0 are introduced into a DLM (suspension or emulsion), then the relaxation time of shear stresses in the medium exceeds the corresponding relaxation time in a pure emulsion or suspension. According to [6], the greater α_0, the greater μ_2 and the smaller G_2, and, hence, there is an increase in λ_1'', and, according to (3.11), of $\tilde{\lambda}_1/\lambda_1'$, i.e., the memory of the medium is retained longer and its elastic properties must manifest themselves more markedly.

Thus, in the present paper a part of the problem of building a rheological model of a cavitating DLM has been solved. A mechanical model has been constructed of a stretched DLM with unlimited growth of cavitation bubbles from the dimensions of nuclei to those of the forming hexagonal cellular structure. A rheological equation in a differential form has been derived describing the relation between shear stresses and strains in a DLM with a fixed volume concentration of cavitation bubbles within the whole range of its realization.

A rheological equation corresponding to the volume stretching of a DLM, i.e., to the case of a monotonically growing volume concentration of cavitation bubbles α_0 must contain rheological coefficients dependent on $\alpha_0(t)$ and on the strain rate. But for these coefficients to be determined, special experimental techniques have to be developed.

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